2007 真题 (数1共)   (2), 4 (10) 11

计算1. 已知矩阵 $A=\begin{bmatrix}3&5&0&0\\1&4&0&0\\0&0&5&1\\0&0&7&3\end{bmatrix}$ $B=\begin{bmatrix}0&0&0&1\\0&0&2&0\\0&1&0&0\\2&0&0&0\end{bmatrix}$ 且 $AXA-\frac{1}{2}B^*A=XA$

(9分) 求矩阵X.

$\because AXA-\frac{1}{2}B^*A=XA$ $|A|\neq 0$ $|B|=4$

有 $AX-\frac{1}{2}B^*=X$ $(A-E)=\begin{pmatrix}2&5&&\\1&3&&\\&&4&1\\&&7&2\end{pmatrix}$ 可逆阵

$AXB-\frac{1}{2}\cdot|B|E=XB$

$(A-E)XB=2E$

$X=2(A-E)^{-1}B^{-1}$

$(A-E\,\vdots\,E)=\left(\begin{array}{cccc|cccc}2&5&&&1&&&\\1&3&&&&1&&\\&&4&1&&&1&\\&&7&2&&&&1\end{array}\right)\to\left(\begin{array}{cccc|cccc}1&3&&&0&1&&\\0&-1&&&1&-2&&\\&&1&\frac{1}{4}&&&\frac{1}{4}&0\\&&0&\frac{1}{4}&&&-\frac{7}{4}&1\end{array}\right)$

$\to\left(\begin{array}{cccc|cccc}1&0&0&0&3&-5&0&0\\0&1&0&0&-1&2&0&0\\0&0&1&0&0&0&2&-1\\0&0&0&1&0&0&-7&4\end{array}\right)=(E\,\vdots\,(A-E)^{-1})$

$(B\,\vdots\,E)=\left(\begin{array}{cccc|cccc}0&0&0&1&1&&&\\0&0&2&0&&1&&\\0&1&0&0&&&1&\\2&0&0&0&&&&1\end{array}\right)\to\left(\begin{array}{cccc|cccc}1&&&&0&0&0&\frac{1}{2}\\&1&&&0&0&1&0\\&&1&&0&\frac{1}{2}&0&0\\&&&1&1&0&0&0\end{array}\right)$

$=(E\,\vdots\,B^{-1})$

$\therefore X=2\begin{pmatrix}3&-5&&\\-1&2&&\\&&2&-1\\&&-7&4\end{pmatrix}\begin{pmatrix}0&0&0&\frac{1}{2}\\0&0&1&0\\0&\frac{1}{2}&0&0\\1&0&0&0\end{pmatrix}$

$=\begin{pmatrix}0&0&-10&3\\0&0&4&-1\\-2&2&0&0\\8&-7&0&0\end{pmatrix}$

线性无关，$AX=0$ 只有零解 $(|A|\neq 0)$

△2. 设n阶方阵A的互不相同的特征值 $\lambda_1,\cdots,\lambda_m$ $(2<m<n)$ 对应的特征向量依次为 $\alpha_1,\cdots,\alpha_m$，证：向量组 $\beta_1=A(\alpha_1+\alpha_2)$，$\beta_2=A(\alpha_1+\alpha_3)$ $\cdots$ $\beta_{m-1}=A(\alpha_1+\alpha_m)$，$\beta_m=A(\alpha_1+\cdots+\alpha_m)$ 线性无关的充要条件是 $\lambda_i\neq 0$ $(i=1,\cdots,m)$

Pf: 由已知 $A\alpha_i=\lambda_i\alpha_i$ $(i=1,\cdots,m)$

$\lambda_i\neq\lambda_j$ 时 可得 α_i 与 α_j 线性无关，则 $\alpha_1,\cdots,\alpha_m$ 线性无关

且 $\beta_1=A(\alpha_1+\alpha_2)=\lambda_1\alpha_1+\lambda_2\alpha_2$

$\beta_2=A(\alpha_1+\alpha_3)=\lambda_1\alpha_1+\lambda_3\alpha_3$

$\cdots$

$\beta_{m-1}=A(\alpha_1+\alpha_m)=\lambda_1\alpha_1+\lambda_m\alpha_m$

$\beta_m=A(\alpha_1+\cdots+\alpha_m)=\lambda_1\alpha_1+\lambda_2\alpha_2+\cdots+\lambda_m\alpha_m$

⟹ 设 $k_1\beta_1+k_2\beta_2+\cdots+k_m\beta_m=0$

有 $(k_1+\cdots+k_m)\lambda_1\alpha_1+(k_1+k_m)\lambda_2\alpha_2+(k_2+k_m)\lambda_3\alpha_3+\cdots+(k_{m-1}+k_m)\lambda_m\alpha_m=0$

活用

⟸ 若 …… $=0$，又有 $\alpha_1,\cdots,\alpha_m$ 线性无关

$\therefore$ 有 ~~$\lambda_1k_1+\lambda_2k_2+\cdots+\lambda_mk_m=0$~~ $\lambda_2k_1+\cdots+\lambda_mk_m=0$

$\therefore\begin{cases}\lambda_1k_1+\lambda_1k_2+\cdots+\lambda_1k_m=0\\\lambda_2k_1+\lambda_2k_m=0\\\lambda_3k_2+\lambda_3k_m=0\\\cdots\\\lambda_mk_{m-1}+\lambda_mk_m=0\end{cases}$ 构造 $AX=0$

$D=\begin{vmatrix}\lambda_1&\cdots&\lambda_1&\lambda_1\\\lambda_2&&0&\lambda_2\\&\ddots&&\vdots\\&&\lambda_m&\lambda_m\end{vmatrix}=-(m-2)\lambda_1\lambda_2\cdots\lambda_m$

活用

$\beta_1,\cdots,\beta_m$ 线性无关 ⟺ $k_1=k_2=\cdots=k_m=0$ 即上述方程只有零解 ⟺ $|D|\neq 0$ ⟺ $\lambda_i\neq 0$