6

5.2) 解：根据 $|\lambda E - A| = 0$ 解得 $\lambda_1 = \lambda_2 = \lambda_3 = 4$ $\lambda_4 = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 4$ 对应的特征向量为 $\alpha_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\alpha_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ $\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

正交化

$\beta_1 = \alpha_1 = (-1, 0, 0, 1)'$ $\eta_1 = \frac{\beta_1}{|\beta_1|} = \frac{1}{\sqrt{2}}(-1, 0, 0, 1)'$

$\beta_2 = \alpha_2 - (\alpha_2, \eta_1)\eta_1 = (0, -1, 0, 1)' - \frac{1}{2}(-1, 0, 0, 1)'$

$= (\frac{1}{2}, -1, 0, \frac{1}{2})'$ $\eta_2 = \frac{\beta_2}{|\beta_2|} = \frac{1}{\sqrt{6}}(1, -2, 0, 1)'$

$\beta_3 = \alpha_3 - (\alpha_3, \eta_1)\eta_1 - (\alpha_3, \eta_2)\eta_2$

$= (0, 0, -1, 1)' - \frac{1}{2}(-1, 0, 0, 1)' - \frac{1}{6}(1, -2, 0, 1)'$

$= (\frac{1}{3}, \frac{1}{3}, -1, \frac{1}{3})'$ $\eta_3 = \frac{\beta_3}{|\beta_3|} = \frac{1}{2\sqrt{3}}(1, 1, -3, 1)'$

$\lambda_4 = 0$ 对应的特征向量为 $\alpha_4 = (1, 1, 1, 1)'$

∵ 不同的特征值对应的特征向量必正交

单位化：$\eta_4 = \frac{\alpha_4}{|\alpha_4|} = \frac{1}{2}(1, 1, 1, 1)'$

∴ 正交阵 $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} \end{bmatrix}$

正交变换 $x = Qy$

$f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (y_1, y_2, y_3, y_4) \begin{bmatrix} 4 & & & \\ & 4 & & \\ & & 4 & \\ & & & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

$= 4y_1^2 + 4y_2^2 + 4y_3^2$

(附) 6. 设 $(x_1, x_2, x_3, x_4)'$ 是数域 K 上矩阵空间 $R^{2\times 2}$ 矩阵 A 在基

$F_1 = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$ $F_2 = \begin{pmatrix} -3 & -3 \\ -3 & -5 \end{pmatrix}$ $F_3 = \begin{pmatrix} 7 & 2 \\ 0 & 0 \end{pmatrix}$ $F_4 = \begin{pmatrix} 11 & 3 \\ 0 & 0 \end{pmatrix}$ 下的坐标

$(y_1, y_2, y_3, y_4)'$ 是 A 的基 G_1, G_2, G_3, G_4 下的坐标，且

$y_1 = 3x_1 - 5x_2$，$y_2 = -x_1 + 2x_2$，$y_3 = 2x_3 + 3x_4$，$y_4 = 5x_3 + 8x_4$

① 求由基 $G_1\ G_2\ G_3\ G_4$ 到基 $F_1\ F_2\ F_3\ F_4$ 的$\boxed{\text{过渡矩阵}}$

② 求$\boxed{\text{基}}$ $G_1\ G_2\ G_3\ G_4$

③ 求矩阵 $B = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$ 在基 $G_1\ G_2\ G_3\ G_4$ 下的$\boxed{\text{坐标}}$

解：① 令 $\xi = (F_1\ F_2\ F_3\ F_4)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (G_1, G_2, G_3, G_4)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

∵ 由已知可得 $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 5 & 8 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

即 $(F_1\ F_2\ F_3\ F_4) = (G_1\ G_2\ G_3\ G_4)A$ 即过渡矩阵得到

②. ∴ $(G_1\ G_2\ G_3\ G_4) = (F_1\ F_2\ F_3\ F_4)A^{-1}$

$(A \vdots E) = \left(\begin{array}{cccc|cccc} 3 & -5 & & & 1 & & & \\ -1 & 2 & & & & 1 & & \\ & & 2 & 3 & & & 1 & \\ & & 5 & 8 & & & & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -2 & & & 0 & -1 & & \\ 0 & 1 & & & 1 & 3 & & \\ & & 1 & \frac{3}{2} & & & \frac{1}{2} & 0 \\ & & 0 & \frac{1}{2} & & & -\frac{5}{2} & 1 \end{array}\right)$

$\to \left(\begin{array}{cccc|cccc} 1 & & & & 2 & 5 & & \\ & 1 & & & 1 & 3 & & \\ & & 1 & & & & 8 & -3 \\ & & & 1 & & & -5 & 2 \end{array}\right) = (E \vdots A^{-1})$ 即 $A^{-1} = \begin{pmatrix} 2 & 5 & & \\ 1 & 3 & & \\ & & 8 & -3 \\ & & -5 & 2 \end{pmatrix}$

解得 $G_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $G_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $G_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $G_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$