

1. 已知 $AX^* - B = 5A$, 其中 X^* 表示矩阵 X 的伴随矩阵, 而

$$A = \begin{bmatrix} 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ 2 & -3 & 0 & 0 \\ -5 & 18 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 3 & -11 \\ 0 & 0 & 2 & -5 \\ -7 & 8 & 0 & 0 \\ 18 & -21 & 0 & 0 \end{bmatrix}$$

解: $\therefore AX^* - B = 5A \quad X X^* = X^* X = |X| E$

$AX^* = 5A + B \quad |A| \neq 0 \quad A A^{-1} = A^{-1} A = E$

$X^* = 5E + A^{-1}B$

$$(A|E) = \begin{bmatrix} 0 & 0 & 3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ -5 & 18 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 3 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 3 & 0 \end{bmatrix}$$

$$\therefore A^{-1}B = \begin{bmatrix} 0 & 0 & 8 & 3 \\ 0 & 0 & 5 & 2 \\ 2 & -5 & 0 & 0 \\ -1 & 3 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 3 & -11 \\ 0 & 0 & 2 & -5 \\ -7 & 8 & 0 & 0 \\ 18 & -21 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 3 & -4 \end{bmatrix} = (E|A^{-1})$$

$$\therefore X^* = 5E + A^{-1}B = \begin{bmatrix} 5 & 0 & 8 & 3 \\ 0 & 5 & 2 & -5 \\ -7 & 8 & 0 & 0 \\ 18 & -21 & 0 & 0 \end{bmatrix} \quad |X^*| = |X|^3 = -64$$

$\therefore X \cdot X^* = |X| E = -4E \quad (X^*)^{-1} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & -\frac{3}{8} \\ 0 & 0 & -\frac{3}{8} & -\frac{1}{8} \end{bmatrix}$

$X = -4(X^*)^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. 设实系数线性方程组 (I) $\begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 2x_1 + x_2 - 6x_3 + 4x_4 = 0 \end{cases}$ 又已知 β_1, β_2 为 (I) 的一个基础解系, 方程组 (II) 与 (I) 有非零公共解, 求全部非零公共解.

解: 将 (I) $\beta = k_1 \beta_1 + k_2 \beta_2$ 代入 (II) 得 $\begin{cases} (4k_1 + k_2) + (2k_1) - 2(k_1 + k_2) + 3(k_1 + k_2) = 0 \\ 2(4k_1 + k_2) + (2k_1) - 6(k_1 + k_2) + 4(k_1 + k_2) = 0 \end{cases}$

即 $\begin{cases} -k_1 + (2+3)k_2 = 0 \\ -2k_1 + (2+2)k_2 = 0 \end{cases}$

方程组 (II) 有非零公共解, 即有 k_1, k_2 不全为零, 则得 $|A| = \begin{vmatrix} -1 & 5 \\ -2 & 4 \end{vmatrix} \neq 0 \quad \therefore d = -4$

② 有 $-k_1 - k_2 = 0$ 即 $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad k \in \mathbb{R}$

公共解为 $\gamma = k \begin{pmatrix} 4 \\ 0 \\ 1 \\ -1 \end{pmatrix} - k \begin{pmatrix} 2 \\ -4 \\ 1 \\ 1 \end{pmatrix} = k \begin{pmatrix} 2 \\ 4 \\ 0 \\ -2 \end{pmatrix}$

③ 设 n 阶方阵 A 满足 $A^k = 0 \quad (k=1, \dots, n-2) \quad A^{n-1} = 0$

④ 证: 满足上述条件的所有 n 阶方阵值域 (即 $\text{Im}(A)$) 是 A 的 $\text{Im}(A)$ 的子集.

⑤ 求 A 的 $\text{Im}(A)$ 的维数.

① $\therefore d(A) = \lambda^{n-1}$
 $\therefore d(A) = \lambda^{n-1} \dots d(A) = \lambda^{n-1}$
 $\therefore d(A) = \lambda^{n-1}$
 $\therefore d(A) = \lambda^{n-1}$