2002 真题(西工大) 4 10 线性代数

特征值 特征向量 特征子空间(不是子空间)

$V=V_1+V_2 \quad V=V_1\oplus V_2$

$(n\geq 2)$

1. 计算n阶行列式 $D_n=\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 2 & 0 & 2 & 0 & \cdots & 0 \\ 3 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ n-1 & 0 & 0 & 0 & \cdots & n-1 \\ n & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix}$

$$D_n=\begin{vmatrix} 1-(n-1) & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & n-1 \\ n-[1+\cdots+(n-1)] & n-1 & n-2 & n-3 & \cdots & 1 \end{vmatrix}=\begin{vmatrix} 2-n & 1 & 1 & \cdots & 1 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & n-1 \\ \frac{n}{2}(3-n) & n-1 & n-2 & \cdots & 1 \end{vmatrix}$$

$$=(2-n)\times(-1)^{1+(n-1)}\cdot(n-1)\cdot(n-1)!$$

$$+(-1)^{1+n}\cdot\frac{n}{2}(3-n)\cdot(n-1)!$$

$$=(-1)^{n+1}(n-1)!\,\left(\frac{n^2-3n+4}{2}\right)$$

2. 设矩阵A的伴随矩阵 $A^*=\begin{bmatrix} 6 & -3 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 \end{bmatrix}$ 且 $3AXA^{-1}=2XA^*+3E$，其中E是单位矩阵，求矩阵X.

$\because A\cdot A^*=A^*A=|A|E \quad |A^*|=||A|A^{-1}|=|A|^3=-27 \quad \therefore |A|=-3$

$A=|A|(A^*)^{-1}$

$$(A^*\vdots E)=\left(\begin{array}{cccc|cccc} 6 & -3 & 0 & 0 & 1 & & & \\ -3 & 3 & 0 & 0 & & 1 & & \\ 0 & 0 & 0 & 1 & & & 1 & \\ 0 & 0 & 3 & -1 & & & & 1 \end{array}\right)\rightarrow\left(\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightarrow\left(\begin{array}{cccc|cccc} 1 & & & & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ & 1 & & & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ & & 1 & & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ & & & 1 & & & & \end{array}\right)=(E\vdots(A^*)^{-1}) \quad \therefore A=\begin{pmatrix} -1 & -1 & & \\ -1 & -2 & & \\ & & -1 & -1 \\ & & & \end{pmatrix}$$

$3AXA^{-1}=2XA^*+3E$

$\Rightarrow 3AX-2|A|X=3A$

$\Rightarrow 3X-2|A|A^{-1}X=3E$

$\Rightarrow (E+2A^{-1})X=E$

$$(A\vdots E)=\left(\begin{array}{cccc|cccc} -1 & -1 & & & 1 & & & \\ -1 & -2 & & & & 1 & & \\ & & -1 & -1 & & & 1 & \\ & & -3 & 0 & & & & 1 \end{array}\right)$$

$$\rightarrow\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)$$

$$\rightarrow\left(\begin{array}{cccc|cccc} 1 & & & & -2 & 1 & & \\ & 1 & & & 1 & -1 & & \\ & & 1 & & & & 0 & -\frac{1}{3} \\ & & & 1 & & & -1 & \frac{1}{3} \end{array}\right)=(E\vdots A^{-1})$$

$$\therefore (E+2A^{-1})=\begin{pmatrix} -3 & 2 & & \\ 2 & -1 & & \\ & & 1 & -\frac{2}{3} \\ & & -2 & \frac{5}{3} \end{pmatrix}$$

$\therefore X=(E+2A^{-1})^{-1}$

$$(E+2A^{-1}\vdots E)=\left(\begin{array}{cccc|cccc} -3 & 2 & & & 1 & & & \\ 2 & -1 & & & & 1 & & \\ & & 1 & -\frac{2}{3} & & & 1 & \\ & & -2 & \frac{5}{3} & & & & 1 \end{array}\right)=\begin{pmatrix} 1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 2 & 1 \end{pmatrix}$$

$$\rightarrow\left(\begin{array}{cccc|cccc} 1 & & & & 1 & 2 & & \\ & 1 & & & 2 & 3 & & \\ & & 1 & & & & 5 & 2 \\ & & & 1 & & & 6 & 3 \end{array}\right)=(E\vdots(E+2A^{-1})^{-1})$$

得 $X=\begin{pmatrix} 1 & 2 & & \\ 2 & 3 & & \\ & & 5 & 2 \\ & & 6 & 3 \end{pmatrix}$

或 $3AXA^{-1}-2XA^*=3E$

$3AX-2X|A|=3A$

$3(A+2E)X=3A$

$X=(A+2E)^{-1}A$

$$(A+2E\vdots E)=\left(\begin{array}{cccc|cccc} 1 & -1 & & & 1 & & & \\ -1 & 0 & & & & 1 & & \\ & & 1 & -1 & & & 1 & \\ & & -3 & 2 & & & & 1 \end{array}\right)\rightarrow\left(\begin{array}{cccc|cccc} 1 & & & & & -1 & & \\ & 1 & & & -1 & -1 & & \\ & & 1 & & & & -2 & -1 \\ & & & 1 & & & -3 & -1 \end{array}\right)=(E\mid(A+2E)^{-1})$$

$$=\begin{pmatrix} -1 & -1 & & \\ -1 & -1 & & \\ & & -2 & -1 \\ & & & \end{pmatrix}\begin{pmatrix} -1 & -1 & & \\ -1 & -2 & & \\ & & -1 & -1 \\ & & & \end{pmatrix}=\begin{pmatrix} 1 & 2 & & \\ 2 & 3 & & \\ & & 5 & 2 \\ & & & \end{pmatrix}$$

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3. 已知n阶方阵 $A=\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$ $(n>1)$ 1) 求A的初等因子 2) 求A的最小多项式

解：A的特征多项式

$$(\lambda E - A) = \begin{pmatrix} \lambda-2 & -1 & \cdots & -1 \\ -1 & \lambda-2 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & \lambda-2 \end{pmatrix}$$

由 $|\lambda E - A| = 0$

得 $\lambda_1 = \cdots = \lambda_{n-1} = 1$

$\lambda_n = n+1$ 　　　∃ 有n个线性无关的特征向量

$$\therefore A \sim J = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & n+1 \end{bmatrix}$$

得 A的初等因子 $\underbrace{\lambda-1,\ \lambda-1,\ \cdots,\ \lambda-1}_{n-1\text{个}}$，$(\lambda-n-1)$

A的最小多项式为 $(\lambda-1)(\lambda-n-1)$

4. 设A与B均为n阶方阵，A有n个互异的特征值，试证：AB=BA 的充要条件是A的特征向量都是B的特征向量。

证："⇒" 若A的特征向量都是B的特征向量

即 $A\xi_i = \lambda_i \xi_i$（λ_i $(i=1,\cdots,n)$ 互异 $\therefore \xi_1 \cdots \xi_n$ 线性无关）

$B\xi_i = \mu_i \xi_i$

有 $AB\xi_i = A(\mu_i \xi_i) = \mu_i(A\xi_i) = \lambda_i \mu_i \xi_i$

$BA\xi_i = B(\lambda_i \xi_i) = \lambda_i(B\xi_i) = \lambda_i \mu_i \xi_i$ 　∴ ~~AB=BA~~

$\therefore \forall \alpha$ 可由 $\xi_1, \cdots, \xi_n$ 线性表示，

∴ 有 $AB\alpha = BA\alpha$ 　　$\therefore AB = BA$.

"⇐" 若 $AB=BA$

设 $A\xi_i = \lambda_i \xi_i$ $(i=1,\cdots,n)$ 则有 $AB\xi_i = BA\xi_i = \lambda_i B\xi_i$ 　故 $B\xi_i \in V_{\lambda_i}$

又 A的n个特征根互异 $\therefore \dim V_{\lambda_i} = 1$

$\therefore \exists \mu_i$ 使 $B\xi_i = \mu_i \xi_i$

$\therefore \xi_1, \cdots, \xi_n$ 都是B的特征向量

（"⇒" $A\xi_i = \lambda_i \xi_i$ 　$B\xi_i = \mu_i \xi_i$ 　∵ A的n个 λ_i 互异 ∴ $\xi_1 \cdots \xi_n$

∴ ∃ 可逆阵 $P = (\xi_1 \cdots \xi_n)$ 使 $P^{-1}AP = \Lambda_1 = \mathrm{diag}(\lambda_1, \cdots$

$P^{-1}BP = \Lambda_2 = \mathrm{diag}(\mu_1, \cdots$

故 $A = P\Lambda_1 P^{-1}$ 　$B = P\Lambda_2 P^{-1}$

$\therefore AB = P\Lambda_1\Lambda_2 P^{-1} = P\Lambda_2\Lambda_1 P^{-1} = BA$ ）

5. 设 n阶方阵 A的非零互异特征值为 $\lambda_1, \cdots, \lambda_m$，其对应的特征向量为 $\beta_1, \cdots, \beta_m$ $(1 \le m < n)$ 齐次线性方程组 $Ax=0$ 的一个基础解系为 $\alpha_1, \cdots, \alpha_r$ $(1 \le r < n)$ 记 $l = \min(m, r)$，证：$\alpha_1, \cdots, \alpha_r, \beta_1+\alpha_1, \cdots, \beta_l+\alpha_l$ 线性无关.

由题意，A的非零……为 $\beta_1, \cdots, \beta_m$

知 $A\beta_i = \lambda_i \beta_i \ (\lambda_i \ne 0)$ 且 $\beta_1 \cdots, \beta_m$ 线性无关.

又由 $Ax=0$ 的基础解系 $\alpha_1, \cdots \alpha_r$

可知 $\alpha_1, \cdots, \alpha_r$ 线性无关 且 $\alpha_1, \cdots, \alpha_r$ 均为 A的特征值零所对应的特征向量.

假设 $l = \min(m, r) = m \le r$

设 $k_1\alpha_1 + \cdots + k_r\alpha_r + k_{r+1}(\beta_1+\alpha_1) + \cdots + k_{r+m}(\beta_m+\alpha_m) = 0$

左右同乘以A，得 $k_{r+1}\lambda_1\beta_1 + \cdots + k_{r+m}\lambda_m\beta_m = 0$

$\because \beta_1, \cdots, \beta_m$ 线性无关 且 $\lambda_i \ (i=1, \cdots, m) \ne 0$

$\therefore k_{r+1} = \cdots = k_{r+m} = 0$

$\therefore k_1\alpha_1 + \cdots + k_r\alpha_r = 0$ $\quad \because \alpha_1, \cdots, \alpha_r$ 线性无关

$\therefore k_1 = \cdots = k_r = 0$

$\therefore \alpha_1, \cdots, \alpha_r, \beta_1+\alpha_1, \cdots, \beta_l+\alpha_l$ 线性无关.

同理 $l = \min(m, r) = r \le m$ 亦成立.

6. 设 $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & c & c \\ 1 & c & 0 & 1 \end{pmatrix}$ (c为实数) $b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ 且齐次线性方程组 $Ax=0$ 的解空间维数为2，求方程组 $Ax=b$ 的解.

解：$\because Ax=0$ 的解空间维数为2

$\therefore \underbrace{r(A) = 4-2=2}$ $\leftarrow$ 4个未知数.

$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & c & c \\ 1 & c & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & c & c \\ 0 & c-2 & -1 & -1 \end{pmatrix}$$ 有 $\frac{1}{c-2} = \frac{c}{-1}$

得 $c=1$

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right) \to \begin{pmatrix} 1 & 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 & -2 \end{pmatrix}$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

得 $Ax=b$ 的解 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$