5. 设 n阶方阵 A的非零互异特征值为 $\lambda_1, \cdots, \lambda_m$, 其对应的特征向量为 $\beta_1, \cdots, \beta_m$ $(1 \le m < n)$ 齐次线性方程组 $AX=0$ 的一个基础解系为 $\alpha_1, \cdots, \alpha_r$ $(1 \le r < n)$ 记 $L=\min(m,r)$, 证: $\alpha_1, \cdots, \alpha_r, \beta_1+\alpha_1, \cdots, \beta_L+\alpha_L$ 线性无关.

由题意 A的非零 … 为 $\beta_1, \cdots, \beta_m$

知 $A\beta_i = \lambda_i \beta_i \ (\lambda_i \neq 0)$ 且 $\beta_1, \cdots, \beta_m$ 线性无关.

又由 $Ax=0$ 的基础解系 $\alpha_1, \cdots, \alpha_r$

可知 $\alpha_1, \cdots, \alpha_r$ 线性无关 且 $\alpha_1, \cdots, \alpha_r$ 均为 A的特征值零所对应的特征向量.

假设 $L=\min(m,r)=m \le r$

设 $k_1\alpha_1 + \cdots + k_r\alpha_r + k_{r+1}(\beta_1+\alpha_1) + \cdots + k_{r+m}(\beta_m+\alpha_m) = 0$

左右同乘以 A, 得 $k_{r+1}\lambda_1\beta_1 + \cdots + k_{r+m}\lambda_m\beta_m = 0$

$\because \beta_1, \cdots, \beta_m$ 线性无关 且 $\lambda_i \ (i=1,\cdots,m) \neq 0$

$\therefore k_{r+1} = \cdots = k_{r+m} = 0$

$\therefore k_1\alpha_1 + \cdots + k_r\alpha_r = 0$ $\quad \because \alpha_1, \cdots, \alpha_r$ 线性无关

$\therefore k_1 = \cdots = k_r = 0$

$\therefore \alpha_1, \cdots, \alpha_r, \beta_1+\alpha_1, \cdots, \beta_L+\alpha_L$ 线性无关.

同理 $L=\min(m,r)=r \le m$ 亦成立.

6. 设 $A=\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & c & c \\ 1 & c & 0 & 1 \end{pmatrix}$ (c为实数) $b=\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ 且齐次线性方程组 $AX=0$ 的解空间维数为2, 求方程组 $AX=b$ 的解.

解: $\because AX=0$ 的解空间维数为2

$\therefore \underbrace{r(A) = 4-2=2}$ $\leftarrow$ 4个未知数.

$(A \vdots b) = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & c & c \\ 1 & c & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & c & c \\ 0 & c-2 & -1 & -1 \end{pmatrix}$ 有 $\frac{1}{c-2} = \frac{c}{-1}$

得 $c=1$

$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right) \to \begin{pmatrix} 1 & 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 & -2 \end{pmatrix}$

$\to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

得 $AX=b$ 的解 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t_1\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$