二次型

4. 设二次曲面 $ax^2+5y^2+3z^2-2xy+6xz-6yz=2$ 经过$\boxed{正交}$变换 $\begin{pmatrix}x\\y\\z\end{pmatrix}=Q\begin{pmatrix}u\\v\\w\end{pmatrix}$ 化为椭圆柱面方程 $4v^2+cw^2=2$，求 a,c 的值及正交矩阵 Q

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解：令 $f(x,y,z)=ax^2+5y^2+3z^2-2xy+6xz-6yz$

$$=(x,y,z)\begin{bmatrix}a&-1&+3\\-1&5&-3\\+3&-3&3\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$=(u,v,w)Q'\begin{bmatrix}a&-1&+3\\-1&5&-3\\+3&-3&3\end{bmatrix}Q\begin{pmatrix}u\\v\\w\end{pmatrix}$$

$$=(u,v,w)\begin{pmatrix}0&&\\&4&\\&&c\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}$$

即 $A=\begin{pmatrix}a&-1&+3\\-1&5&-3\\+3&-3&3\end{pmatrix}\sim\begin{pmatrix}0&&\\&4&\\&&c\end{pmatrix}=B$

由 $|\lambda E-A|=|\lambda E-B|$ 得 $\begin{cases}a=5\\c=9\end{cases}$

$\therefore A=\begin{pmatrix}5&-1&3\\-1&5&-3\\3&-3&3\end{pmatrix}$ 当 $\lambda_1=0$ 时 $\alpha_1=(-1,1,2)'$

$\lambda_2=4$ 时 $\alpha_2=(1,1,0)'$

$\lambda_3=9$ 时 $\alpha_3=(1,-1,1)'$

$\because \lambda_1,\lambda_2,\lambda_3$ 各不相同 $\therefore$ 对应的特征向量相互正交.

单位化：$\beta_1=\frac{1}{\sqrt6}(-1,1,2)'$

$\beta_2=\frac{1}{\sqrt2}(1,1,0)'$

$\beta_3=\frac{1}{\sqrt3}(1,-1,1)'$

$\therefore$ 正交矩阵 $Q=\begin{bmatrix}-\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&-\frac{1}{\sqrt3}\\\frac{2}{\sqrt6}&0&\frac{1}{\sqrt3}\end{bmatrix}$

使 $f=4v^2+9w^2=2$

5. 已知 $R^{2\times2}$ 线性变换 $T(X)=\begin{bmatrix}3x_1+x_2+x_3-x_4 & x_1+3x_2-x_3+x_4\\x_1-x_2+3x_3+x_4 & -x_1+x_2+x_3+3x_4\end{bmatrix}$

$\forall X=\begin{bmatrix}x_1&x_2\\x_3&x_4\end{bmatrix}$ 1) 证 T 是对称变换

2) 求 $R^{2\times2}$ 的一组标准正交基，使 T 在该基下的矩阵为对角矩阵.

解：① 线性变换 T 在 $E_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $E_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $E_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $E_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ 下的矩阵为 A.

即 $T(E_{11},E_{12},E_{21},E_{22})=(E_{11},E_{12},E_{21},E_{22})A$ 得 $A=\begin{pmatrix}3&1&1&-1\\1&3&-1&1\\1&-1&3&1\\-1&1&1&3\end{pmatrix}$

$\because A'=A$ $\therefore T$ 为对称变换.

②. 根据 $|\lambda E-A|=0$

得 $\lambda_1=0$ 特征向量 $\alpha_1=(1,-1,-1,1)'$ $\gamma_1=\frac{1}{2}(1,-1,-1,1)'$

$\lambda_2=\lambda_3=\lambda_4=4$ 特征向量 $\alpha_2=(1,1,0,0)'$

$\alpha_3=(1,0,1,0)'$

$\alpha_4=(1,0,0,-1)'$

正交化 $\beta_2=\alpha_2=(1,1,0,0)'$ $\gamma_2=\frac{1}{\sqrt2}(1,1,0,0)'$

$\beta_3=\alpha_3-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=(\frac12,-\frac12,1,0)'$ $\gamma_3=\frac{1}{\sqrt6}(1,-1,2,0)'$

$\beta_4=\alpha_4-\frac{(\alpha_4,\beta_2)}{(\beta_2,\beta_2)}\beta_2-\frac{(\alpha_4,\beta_3)}{(\beta_3,\beta_3)}\beta_3$

$=(\frac13,-\frac13,-\frac13,-1)'$

$\gamma_4=\frac{\sqrt3}{2}(\frac13,-\frac13,-\frac13,-1)'=\frac{1}{2\sqrt3}(1,-1,-1,-3)'$

$\therefore P=\begin{bmatrix}\frac12&\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{2\sqrt3}\\-\frac12&\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}&-\frac{1}{2\sqrt3}\\-\frac12&0&\frac{2}{\sqrt6}&-\frac{1}{2\sqrt3}\\\frac12&0&0&-\frac{3}{2\sqrt3}\end{bmatrix}$ 有 $(B_{11},B_{12},B_{21},B_{22})=(E_{11},E_{12},E_{21},E_{22})P$

得标准正交基 $B_{11}=\frac12\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$ $B_{12}=\frac{1}{\sqrt2}\begin{pmatrix}1&1\\0&0\end{pmatrix}$

$B_{21}=\frac{1}{\sqrt6}\begin{pmatrix}1&-1\\2&0\end{pmatrix}$ $B_{22}=\frac{1}{2\sqrt3}\begin{pmatrix}1&-1\\-1&-3\end{pmatrix}$