7. 已知三维空间 $V$ 的一组基 $\alpha_1, \alpha_2, \alpha_3$，且该基的度量矩阵 $A=\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

1) 计算欧氏内积 $(\alpha_1+\alpha_2, \alpha_1)$，$(\alpha_2, \alpha_3)$，$(\alpha_1+2\alpha_2-\alpha_3, 2\alpha_2+\alpha_3)$

2) 求 $V$ 的一组正交基 $\beta_1, \beta_2, \beta_3$ 和标准正交基 $\gamma_1, \gamma_2, \gamma_3$.

解 ① 由度量矩阵 $A=\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ 可知 $(\alpha_1,\alpha_1)=1$ $(\alpha_2,\alpha_2)=2$ $(\alpha_3,\alpha_3)=4$ $(\alpha_1,\alpha_2)=-1$ $(\alpha_1,\alpha_3)=1$ $(\alpha_2,\alpha_3)=0$.

17

$\therefore (\alpha_1+\alpha_2, \alpha_1) = (\alpha_1,\alpha_1)+(\alpha_2,\alpha_1) = 0$

$(\alpha_2, \alpha_3) = 0$

$(\alpha_1+2\alpha_2-\alpha_3, 2\alpha_2+\alpha_3) = 2(\alpha_1,\alpha_2)+(\alpha_1,\alpha_3)+4(\alpha_2,\alpha_2)+2(\alpha_2,\alpha_3)$
$-2(\alpha_3,\alpha_2)-(\alpha_3,\alpha_3)$
$=3$

② 法1) 根据 $|\lambda E-A|=0$ 得 $\lambda_i=$ 不易求

法2) $\beta_1=\alpha_1$ $(\beta_1,\beta_1)=1$

$\beta_2=\alpha_2-\dfrac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\alpha_1+\alpha_2$ $(\beta_2,\beta_2)=1$

$\beta_3=\alpha_3-\dfrac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\dfrac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=-2\alpha_1-\alpha_2+\alpha_3$

$(\beta_3,\beta_3)=2$

$\therefore \gamma_1=\beta_1=\alpha_1$

$\gamma_2=\beta_2=\alpha_1+\alpha_2$

$\gamma_3=\dfrac{\beta_3}{|\beta_3|}=\dfrac{\sqrt{2}}{2}(-2\alpha_1-\alpha_2+\alpha_3)$

8. 已知二次型 $f(x_1, x_2, x_3)=x_1^2+ax_2^2+x_3^2+2x_1x_2+2ax_1x_3+2x_2x_3$ 的秩为2. 1) 试求一正交变换化二次型 $f$ 为标准形

2) $f=1$ 为何种二次曲面

解：1) $f(x_1,x_2,x_3)=\cdots=(x_1,x_2,x_3)A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ 即 $A=\begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$ 的秩为2

$A=\begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & a \\ 0 & a-1 & 1-a \\ 0 & 1-a & 1-a^2 \end{pmatrix}$ 得 $a=-2$

$|\lambda E-A|=\begin{vmatrix} \lambda-1 & -1 & 2 \\ -1 & \lambda+2 & -1 \\ 2 & -1 & \lambda-1 \end{vmatrix}=0$ 解得 $\lambda_1=0$ 对应的 $\alpha_1=(1,1,1)'$

$\lambda_2=3$ $\alpha_2=(1,0,-1)'$

$\lambda_3=-3$ $\alpha_3=(1,-2,1)'$

$\because$ 特征值互异 $\therefore$ 特征向量必相互正交.

单位化. $\gamma_1=\frac{1}{\sqrt{3}}(1,1,1)'$ $\gamma_2=\frac{1}{\sqrt{2}}(1,0,-1)'$ $\gamma_3=\frac{1}{\sqrt{6}}(1,-2,1)'$

得 $Q=\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$ 通过 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=Q\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

得 $f=3y_2^2-3y_3^2$

2) $f=3y_2^2-3y_3^2=1$ 为双曲柱面