第四章 态和力学量表象

§4.1 态的表象

量子力学中态和力学量的具体表示方式称为表象，以前所采用的表象是坐标表象，本章将讨论其他的表象。

（1）动量表象

动量本征函数

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

组成完全系，任意态Ψ可以由它展开

$$\Psi(x,t) = \int C(p,t)\psi_p(x)dp$$

$$C(p,t) = \int \psi_p^*(x)\Psi(x,t)dx$$

如果 **Ψ(x,t)** 已经归一化，则**C(p,t)**也已经归一化.

证明

$$1 = \int \Psi^*(x,t)\Psi(x,t)dx$$

$$= \int [\int C(p',t)\psi_{p'}(x)dp']^*[\int C(p,t)\psi_p(x)dp]dx$$

$$= \int\int C(p',t)^* C(p,t)[\int \psi_{p'}^*(x)\psi_p(x)dx]dp'dp$$

$$= \int \int C(p',t)^* C(p,t)\delta(p'-p)dp'dp$$

$$= \int C(p,t)^* C(p,t)dp$$

Ψ (r,t)是波函数的坐标表象。 C(p, t)是波函数的动量表象。它们为可逆的傅里叶变换。它们是同一个量子态的两种描述。坐标表象描述的是体系处于该态时，测量粒子位置所得结果在x → x+dx范围内的概率，动量表象描述的是体系处于该态时，测量粒子动量所得结果在p → p+dp范围内的概率.前者描述坐标与概率关系，后者描述动量与概率关系。但二者描写的是同一状态。

具有确定动量 **p'**的自由粒子的状态用波函数 **Ψ(x,t)**描写，则该状态的动量表象为

$$\Psi(x,t)=\psi_{p'}(x)e^{-iE_{p'}t/\hbar}$$

$$E_{p'}=\frac{p'^2}{2\mu}$$

$$C(p,t)=\int \psi_p^*(x)\Psi(x,t)dx=\int \psi_p^*(x)\psi_{p'}(x)e^{-iE_{p'}t/\hbar}dx$$

$$=e^{-iE_{p'}t/\hbar}\int \psi_p^*(x)\psi_{p'}(x)dx=e^{-iE_{p'}t/\hbar}\delta(p-p')$$

所以，在动量表象中，粒子具有确定动量**p'**的波函数是以动量**p**为变量的**δ**函数

同理

坐标**x**在自身表象中对应于确定值**x'**的本征函数是**δ(x'-x)**,因为

$$x\delta(x'-x)=x'\delta(x'-x)$$

所以 $$\psi_{x'}(x)=\delta(x'-x)$$

（2）力学量表象

（2.1）分立谱情形

假定算符Q的本征值是 Q_1, Q_2, ..., Q_n, ...,

相应的本征函数是 $u_1(x)$, $u_2(x)$, ..., $u_n(x)$, ...。

Ψ(x, t) 由Q的本征函数展开：

$$\Psi(x,t)=\sum_n a_n(t)u_n(x)$$

$$a_n(t)=\int u_n^*(x)\Psi(x,t)dx$$

如果 Ψ, u_n 已经归一化，则$a_n(t)$ 也已经归一化

证明：

$a_1(t)$, $a_2(t)$, ..., $a_n(t)$, ... 是对同样状态Ψ(x,t)的Q表象表达

$$1=\int \Psi^*(x,t)\Psi(x,t)dx$$

$$=\int [\sum_m a_m(t)u_m(x)]^*\sum_n a_n(t)u_n(x)dx$$

$$=\sum_m\sum_n a_m^*(t)a_n(t)\int u_m^*(x)u_n(x)dx$$

$$=\sum_m\sum_n a_m^*(t)a_n(t)\delta_{mn}$$

$$=\sum_n a_n^*(t)a_n(t)$$

可写为矩阵形式

$|a_n|^2$ 表示在状态Ψ(x,t)时，测量力学量Q的结果为Q_n的概率

$$\Psi = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \end{pmatrix} \qquad \Psi^{+} = \begin{pmatrix} a_1(t)^* & a_2(t)^* & \cdots & a_n(t)^* & \cdots \end{pmatrix}$$

共轭矩阵

$$\Psi^{+}\Psi = \begin{pmatrix} a_1(t)^* & a_2(t)^* & \cdots & a_n(t)^* & \cdots \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \end{pmatrix}$$

$$= \sum_n a_n(t)^* a_n(t) = 1$$

（2.2）连续谱情形

假定算符Q的本征值和本征函数是

Q_1, Q_2, ..., Q_n, ..., q

$u_1(x)$, $u_2(x)$, ..., $u_n(x)$, ..., $u_q(x)$

氢原子的能量就是这样一个力学量，既有分立本征值，还有连续本征值

$$\Psi(x,t)=\sum_n a_n(t)u_n(x)+\int a_q(t)u_q(x)dq$$

$$\sum_n a_n^*(t)a_n(t)+\int a_q^*(t)a_q(t)dq=1$$

$$\begin{cases} a_n(t)=\int u_n^*(x)\Psi(x,t)dx \\ a_q(t)=\int u_q^*(x)\Psi(x,t)dx \end{cases}$$

$|a_n(t)|^2$ --- Q_n--- Q---$\Psi(x,t)$

$|a_q(t)|^2dq$ --- $q \rightarrow q + dq$ ---Q---$\Psi(x,t)$

$$\Psi=\begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \\ a_q(t) \end{pmatrix}$$

$$\Psi^+=\begin{pmatrix} a_1(t)^* & a_2(t)^* & \cdots & a_n(t)^* & \cdots & a_q(t)^* \end{pmatrix}$$

$$\Psi^+\Psi=1$$

（3）讨论

	坐标表象	动量表象
动量本征函数	$\Psi_{p'}(x,t)=[1/(2\pi\hbar)]^{1/2}\exp[i(p'x-E't)/\hbar]$	$C(p,t)=\delta(p'-p)\exp[-iE't/\hbar]$
不含时间的动量本征函数	$\psi_{p'}(x)=[1/(2\pi\hbar)]^{1/2}\exp[ip'x/\hbar]$	$C(p)=\delta(p'-p)$
本征方程	$p\,\psi_{p'}(x)=p'\psi_{p'}(x)$	$p\delta(p'-p)=p'\delta(p'-p)$

量子力学中的表象		坐标体系
波函数的不同表象		不同坐标体系的一组坐标分量
$u_1(x), u_2(x), \ldots, u_n(x), \ldots$ 基矢		i, j, k,
$a_1(t), a_2(t), \ldots, a_n(t), \ldots$	→	A_x, A_y, A_z
量子态 $\Psi(x,t)$ 希尔伯特（Hilbert）空间的态矢量		矢量 A

Q 表象中的基矢量

希尔伯特空间

$$\mathbf{u_1(x), \; u_2(x), \; ..., \; u_n(x), \; ...}$$

$$\Psi = \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_n(t) \\ \vdots \end{pmatrix}$$

§4.2　算符的矩阵表示

（1）力学量的矩阵表示

（2）算符F在表象Q中的性质

（3）力学量Q具有连续本征值的情形

（1）力学量的矩阵表示

分立本征值，

坐标 X 表象 | Q表象 | Ψ用$\{u_n(x)\}$展开

$$\Phi(x,t)=\hat{F}(x,\hat{p})\Psi(x,t)=\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})\Psi(x,t)$$

$$\begin{cases}\Psi(x,t)=\sum_m a_m(t)u_m(x)\\ \Phi(x,t)=\sum_m b_m(t)u_m(x)\end{cases}$$

$$\sum_m b_m(t)u_m(x)=\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})\sum_m a_m(t)u_m(x)$$

$u^*_n(x)\times$ 并 ∫ dx

$$\sum_m b_m(t)\int u_n^* u_m(x)dx=\sum_m [\int u_n^*\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})u_m(x)dx]a_m(t)$$

$$\sum_m b_m(t)\delta_{nm}=\sum_m F_{nm}a_m(t)$$

$$b_n(t)=\sum_m F_{nm}a_m(t)$$

Q 表象

$$F_{nm}\equiv\int u_n^*(x)\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})u_m(x)dx$$

Q		X
$\{A_m(t)\}$		$\Psi(x,t)$
$\{B_n(t)\}$	$\longrightarrow$	$\Phi(x,t)$
H_{nm} F_{nm}		$\hat{H}$ F

$$b_n(t) = \sum_m F_{nm} a_m(t) \qquad n = 1, 2, \cdots$$

$$\begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1m} & \cdots \\ F_{21} & F_{22} & \cdots & F_{2m} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & \mathrm{F}_{\mathrm{nm}} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ a_m(t) \\ \vdots \end{pmatrix} \Longrightarrow \mathbf{\Phi = F\Psi}$$

（2）算符F在表象Q中的性质

1）厄米算符F的厄米矩阵

$$F_{nm}=\int u_n^*(x)\hat{F}u_m(x)dx$$

$$=[\int u_n(x)(\hat{F}u_m(x))^*dx]^*$$

$$=[\int u_m^*(x)\hat{F}u_n(x)dx]^*$$

$$=F_{mn}^*=(\tilde{F})^*_{nm}=(F^+)_{nm}$$

表明：厄米算符对应的矩阵是厄米矩阵

2）算符在自身表象Q中的形式

$$\hat{Q}u_n(x)=Q_nu_n(x)$$

$$Q_{nm}=\int u_n^*(x)\hat{Q}u_m(x)dx$$

$$=Q_m\int u_n^*(x)u_m(x)dx=Q_m\delta_{nm}$$

$$Q=\begin{pmatrix} Q_1 & 0 & \cdots & \cdots & \cdots & \cdots\cdots \\ 0 & Q_2 & 0 & \cdots & \cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ 0 & 0 & \cdots & \cdots & \cdots Q_n & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\cdots \end{pmatrix}$$

力学量Q的矩阵在自身表象中是对角阵，对角元素是其本征值

（3）力学量Q具有连续本征值的情形

分立谱 → 连续谱

$$u_n^*(x),\ \ u_m(x) \ \rightarrow\ u_q^*(x),\ \ u_q(x)$$

$$a_n(t),\ \ b_m(t) \ \rightarrow\ a_q(t), b_q(t)$$

$$\sum_n \ \rightarrow\ \int dq$$

F在 Q表象中的矩阵元

$$F_{qq'} = \int u_q^*(x)\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})u_{q'}(x)dx$$

连续矩阵

例题: 在坐标X表象中找到F的矩阵元

$$F_{xx'}=\int\delta(x-x'')\hat{F}(x'',-i\hbar\tfrac{\partial}{\partial x''})\delta(x'-x'')dx'' =\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})\delta(x'-x)$$

例题: 在动量p表象中找到F的矩阵元

$$F_{pp'}=\int\psi_p{}^*(x)\hat{F}(x,-i\hbar\tfrac{\partial}{\partial x})\psi_{p'}(x)dx$$

1) $\hat{F}=\hat{p}$

$$p_{pp'}=\int\psi_p{}^*(x)\hat{p}\psi_{p'}(x)dx$$

$$=p'\int\psi_p{}^*(x)\psi_{p'}(x)dx$$

$$=p'\delta(p-p')$$

2) $\hat{F}=x$

$$x_{pp'}=\int\psi_p{}^*(x)x\psi_{p'}(x)dx$$

$$=\frac{1}{\sqrt{2\pi\hbar}}\int[e^{-ipx/\hbar}x]\psi_{p'}(x)dx$$

$$=\frac{1}{\sqrt{2\pi\hbar}}\int(i\hbar\tfrac{\partial}{\partial p})e^{-ipx/\hbar}\psi_{p'}(x)dx$$

$$=(i\hbar\tfrac{\partial}{\partial p})\int\psi_p{}^*(x)\psi_{p'}(x)dx$$

$$=(i\hbar\tfrac{\partial}{\partial p})\delta(p-p')$$

§4.3 量子力学公式的矩阵表达

（1）平均值公式

（2）本征函数

（3）薛定谔方程的矩阵形式

（1）平均值公式

坐标X表象

$$\overline{F}=\int \Psi^*(x,t)\hat{F}\Psi(x,t)dx$$

力学量Q表象

$$\begin{cases}\Psi(x,t)=\sum_n a_n(t)u_n(x)\\ \Psi^*(x,t)=\sum_n a_n^*(t)u_n^*(x)\end{cases}$$

$$\overline{F}=\int \sum_m a_m^*(t)u_m^*(x)\hat{F}\sum_n a_n(t)u_n(x)dx$$

$$=\sum_m\sum_n a_m(t)^*[\int u_m^*(x)\hat{F}u_n(x)dx]a_n(t)$$

$$=\sum_m\sum_n a_m(t)^*F_{mn}a_n(t)$$

$$\overline{F}=(a_1^*(t),a_2^*(t),\cdots,a_m^*(t)\cdots)\begin{pmatrix}F_{11} & F_{12} & \cdots & F_{1n} & \cdots\\ F_{21} & F_{22} & \cdots & F_{2n} & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ F_{m1} & F_{m2} & \cdots & F_{mn} & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\end{pmatrix}\begin{pmatrix}a_1(t)\\ a_2(t)\\ \vdots\\ a_n(t)\\ \vdots\end{pmatrix}$$

$$\overline{F}=\Psi^+F\Psi$$

（2）本征函数

$$\hat{F}\psi(x)=\lambda\psi(x) \qquad F\psi=\lambda\psi$$

$$\begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1n} & \cdots \\ F_{21} & F_{22} & \cdots & F_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix}=\lambda\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} F_{11}-\lambda & F_{12} & \cdots & F_{1n} & \cdots \\ F_{21} & F_{22}-\lambda & \cdots & F_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn}-\lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix}=0$$

线性齐次代数方程组 $\sum_n (F_{mn}-\lambda\delta_{mn})a_n=0$

$$m=1,2,\cdots$$

$$\begin{vmatrix} F_{11}-\lambda & F_{12} & \cdots & F_{1n} & \cdots \\ F_{21} & F_{22}-\lambda & \cdots & F_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn}-\lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0$$

久期方程

方程解 λ_1, λ_2, ..., λ_i , ...λ_n, ... 是F的本征值，对于 λ_i,有函数

$$\Psi_i=\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \\ \vdots \end{pmatrix},\ i=1,2,\cdots,n\cdots$$

（3）薛定谔方程的矩阵形式

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=\hat{H}\Psi(x,t) \qquad \Psi(x,t)=\sum_n a_n(t)u_n(x)$$

Q 表象

$$i\hbar\frac{\partial}{\partial t}\sum_n a_n(t)u_n(x)=\hat{H}\sum_n a_n(t)u_n(x)$$

u_m*(t) × 且 ∫dx

$$i\hbar\frac{\partial}{\partial t}\sum_n a_n(t)\int u_m^*(x)u_n(x)dx=\sum_n a_n(t)\int u_m^*(x)\hat{H}u_n(x)dx$$

$$i\hbar\frac{\partial}{\partial t}\sum_n a_n(t)\delta_{mn}=\sum_n a_n(t)H_{mn} \qquad H_{mn}=\int u_m^*(x)\hat{H}u_n(x)dx$$

$$i\hbar\frac{\partial}{\partial t}a_m(t)=\sum_n H_{mn}a_n(t)$$

$$m,n=1,2,\cdots$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}a_1(t)\\a_2(t)\\\vdots\\a_n(t)\\\vdots\end{pmatrix}=\begin{pmatrix}H_{11}&H_{12}&\cdots&H_{1n}&\cdots\\H_{21}&H_{22}&\cdots&H_{2n}&\cdots\\\cdots&\cdots&\cdots&\cdots&\cdots\\H_{m1}&H_{m2}&\cdots&H_{mn}&\cdots\\\cdots&\cdots&\cdots&\cdots&\cdots\end{pmatrix}\begin{pmatrix}a_1(t)\\a_2(t)\\\vdots\\a_n(t)\\\vdots\end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi$$

作业

4.1

4.3

4.4

§4.4　幺正变换

（1）不同表象之间的变换和幺正变换矩阵

（2）波函数和算符的变换关系

（3）幺正变换的性质

（1）不同表象之间的变换和幺正变换矩阵

（i）幺正变换矩阵

$$\hat{A}\psi_n(x) = A_n\psi_n(x) \qquad n = 1,2,\ldots$$

$$\hat{B}\varphi_\beta(x) = B_\beta\varphi_\beta(x) \qquad \beta = 1,2,\ldots$$

算符A和B的本征方程
和正交归一的本征函数系

$$F_{nm} = \int \psi_m{}^*(x)\hat{F}\psi_n(x)dx, \quad m,n = 1,2,\ldots$$

$$F'_{\alpha\beta} = \int \varphi_\alpha{}^*(x)\hat{F}\varphi_\beta(x)dx, \quad \alpha,\beta = 1,2,\ldots$$

$$\varphi_\beta(x) = \sum_n S_{n\beta}\psi_n(x)$$

将B的基矢按A的完全系展开

$$\varphi_\alpha^*(x) = \sum_m \psi_m^*(x)S_{m\alpha}^*$$

$$S_{n\beta} = \int \psi_n^*(x)\varphi_\beta(x)dx$$

$$S_{m\alpha}^* = \int \psi_m^*(x)\varphi_\alpha^*(x)dx$$

矩阵形式

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_\beta(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} S_{11} & S_{21} & \cdots & S_{n1} & \cdots \\ S_{12} & S_{22} & \cdots & S_{n2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{1\beta} & S_{2\beta} & \cdots & S_{n\beta} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_n(x) \\ \vdots \end{pmatrix} \Rightarrow \begin{matrix} \Phi = \tilde{S}\Psi \\ \Phi^+ = \tilde{\Phi}^* = \Psi^+ S^* \end{matrix}$$

（ｉｉ） S矩阵的么正性质

a）$S^+ S = I$

$$\delta_{\alpha\beta} = \int \varphi_\alpha{}^*(x)\varphi_\beta(x)dx = \sum_{nm}\int \psi_m^*(x)S_{m\alpha}^* S_{n\beta}\psi_n(x)dx = \sum_{nm} S_{m\alpha}^* S_{n\beta}\int \psi_m^*(x)\psi_n(x)dx$$

$$= \sum S_{m\alpha}^* S_{n\beta}\delta_{mn} = \sum_m S_{m\alpha}^* S_{m\beta} = \sum_m (S^+)_{\alpha m} S_{m\beta} = (S^+ S)_{\alpha\beta}$$

b）$S S^+ = I$

$$(SS^+)_{jk} = \sum_\alpha S_{j\alpha}(S^+)_{\alpha k} = \sum_\alpha S_{j\alpha}(\tilde{S}^*)_{\alpha k} = \sum_\alpha S_{j\alpha} S_{k\alpha}^*$$

$$= \sum_\alpha \int \psi_j^*(x)\varphi_\alpha(x)dx\left[\int \psi_k^*(x')\varphi_\alpha(x')dx'\right]^* = \sum_\alpha \int \psi_j^*(x)\varphi_\alpha(x)dx \int \varphi_\alpha^*(x')\psi_k(x')dx'$$

$$= \sum_\alpha \int \psi_j^*(x)\varphi_\alpha(x)dx \bullet c_\alpha = \int \psi_j^*(x)\bullet\left[\sum_\alpha c_\alpha\varphi_\alpha(x)\right]dx = \int \psi_j^*(x)\psi_k(x)dx = \delta_{jk}$$

综上 $S^{+}S = SS^{+} = I \rightarrow S^{+} = S^{-1} \rightarrow$ S为么正矩阵

(iii) 么正矩阵的计算方法

方法一：计算所有的矩阵元

$$S_{k\beta} = \int \psi_k^*(x)\varphi_\beta(x)dx$$

方法二：计算基矢的变换系数

$$\varphi_\beta = \sum_k S_{k\beta}\psi_k \qquad \varphi_\beta = \begin{pmatrix} S_{1\beta} \\ S_{2\beta} \\ \vdots \\ S_{k\beta} \\ \vdots \end{pmatrix} \text{ A 表象}$$

假定A和B本征矢的数目是3，即ψ_1, ψ_2, ψ_3和$\varphi_1, \varphi_2, \varphi_3$.

若在A表象中已知 φ_β, $(\beta = 1, 2, 3)$

$$\varphi_1 = S_{11}\psi_1 + S_{21}\psi_2 + S_{31}\psi_3$$

$$\varphi_2 = S_{12}\psi_1 + S_{22}\psi_2 + S_{32}\psi_3$$

$$\varphi_3 = S_{13}\psi_1 + S_{23}\psi_2 + S_{33}\psi_3$$

则在A表象中B的本征矢是

$$\phi_1 = \begin{pmatrix} S_{11} \\ S_{21} \\ S_{31} \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} S_{12} \\ S_{22} \\ S_{32} \end{pmatrix} \qquad \phi_3 = \begin{pmatrix} S_{13} \\ S_{23} \\ S_{33} \end{pmatrix}$$

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$

A → **B** 的幺正变换矩阵

$$\Phi = \tilde{S}\Psi$$

（2）波函数和算符的变换关系

（a）波函数（态矢量）的变换关系

对任意态**u** $u=\sum_k a_k\psi_k=\sum_k b_k\varphi_k$

在**A**表象中**u**可写为

$$u=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \end{pmatrix}\equiv a$$

在**B**表象中**u**可写为

$$u=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_\alpha \\ \vdots \end{pmatrix}\equiv b$$

$$b_\alpha(t)=\int\varphi_\alpha^*(x)u(x,t)dx=\int[\sum_m\psi_m^*(x)S_{m\alpha}^*]u(x,t)dx$$

$$=S_{m\alpha}^*\sum_m\int\psi_m^*(x)u(x,t)dx=S_{m\alpha}^*\sum_m a_m(t)=\sum_m S_{m\alpha}^*a_m(t)$$

$$=\sum_m S_{\alpha m}^+a_m(t)=(S^+a)_\alpha$$

→ **b = S⁺ a= S⁻¹ a**

$$b=S^+a=S^{-1}a$$

（b）算符F的变换关系

$$\varphi_\beta(x)=\sum_n S_{n\beta}\psi_n(x)$$

$$\varphi_\alpha^*(x)=\sum_m \psi_m^*(x)S_{m\alpha}^*$$

A： $F_{jk}=\int\psi_j^*(x)\hat{F}\psi_k(x)dx$

B：

$$F'_{\alpha\beta}=\int\varphi_\alpha^*(x)\hat{F}\varphi_\beta(x)dx=\int[\sum_m\psi_m^*(x)S_{m\alpha}^*]\hat{F}[\sum_n S_{n\beta}\psi_n(x)]dx$$

$$=\sum_{mn}\int\psi_m^*(x)S_{m\alpha}^*\hat{F}S_{n\beta}\psi_n(x)dx=\sum_{mn}S_{m\alpha}^*\int\psi_m^*(x)\hat{F}\psi_n(x)dxS_{n\beta}$$

$$=\sum_{mn}S_{m\alpha}^*F_{mn}S_{n\beta}=\sum_{mn}S_{\alpha m}^+F_{mn}S_{n\beta}=(S^+FS)_{\alpha\beta}$$

$$\mathbf{F'=S^{+}\,F\,S=S^{-1}\,F\,S}$$

（3）幺正变换的性质

（a）本征值不变

F 在A表象中： F a = λa

在中： **B表象 F' b =** $S^{-1} F S S^{-1} a = S^{-1} F a = S^{-1} \lambda a$

$= \lambda S^{-1} a = \lambda b$

（b）矩阵的迹不变

矩阵的迹 $SpF = \sum_k F_{kk}$

$$Sp(F') = Sp(S^{-1}FS)$$
$$= Sp(SS^{-1}F)$$
$$= Sp(F)$$

作业

4.5

§4.5　狄拉克符号

(1)　引论

(2)　态矢量

(3)　算符

(4)　小结

(1）引论

- 表象： 坐标X，力学量Q
- 状态矢量**A**不涉及具体表象**(A_x, A_y, A_z)**
- 量子力学中描写态和力学量可以不用具体表象，这种描写的方式是狄拉克最先引用的，这样的一套符号就称为狄拉克符号。

(2) 态矢量

（a）右矢 $|\,>$

线性谐振子状态$\psi_n(x)$的量子数n;氢原子状态$\psi_{nlm}(r,\theta,\varphi)$ 的量子数 n,l,m可用于标记体系状态，如下所示

$$\psi_n(x) \to |\psi_n> \to |n>\,;\ \psi_{nlm} \to |\psi_{nlm}> \to |n,l,m>$$

本征态 $|x>$, $|p>$, $|Q_n>$ ...

完全右矢集合(或基组)，基矢

如

$$|\psi>=\sum_n a_n |n>$$

（b）左矢 <|

狄拉克符号

左矢	右矢
<n |	|n >
< n,l,m |	|n,l,m >
<x' |	|x' >
<A |	|A >
< l,m |	|l,m >
<p' |	|p' >
<Qn |	|Qn >
bra	ket

$< | = (| >)^{+}$ 共轭矢量

<ψ | 和|ψ> 共轭矢量，<p' |, <x' |, <Q_n | 组成左矢完全系（基组）

$$\langle\psi|\varphi\rangle \equiv (\psi,\varphi) = \int \psi^*(x)\varphi(x)dx = (\langle\psi|)\bullet(|\varphi\rangle)$$

$$\langle\psi|\varphi\rangle^* \equiv (\psi,\varphi)^* = (\varphi,\psi) = \langle\varphi|\psi\rangle^{+}$$

$\langle\psi|\varphi\rangle = 0$，正交；$\langle\psi|\psi\rangle = 1$，归一

本征方程 $\hat{F}|n\rangle = f_n|n\rangle$，$\langle n|n'\rangle = \delta_{nn'}$ 正交归一

$\{|n\rangle\}$ 构成F表象希尔伯特空间基矢，则

任意态矢在 F表象展开（完全性）

$$|\psi> = \sum_n a_n |n>$$

F表象

$$\langle n'|\psi\rangle = \sum_n a_n \langle n'|n\rangle = \sum_n a_n \delta_{n'n} = a_{n'}$$

$$|1\rangle = \begin{pmatrix} a_1 = 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 0 \\ a_2 = 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \cdots\cdots |n\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_n = 1 \\ \vdots \\ 0 \end{pmatrix} \cdots\cdots$$

$$a_n = \langle n|\psi\rangle$$

$$\{a_n\} = \{\langle n|\psi\rangle\}$$

↓

$$|\psi\rangle = \begin{pmatrix} <1|\psi> \\ <2|\psi> \\ \vdots \\ <n|\psi> \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots$$

$$= \sum_n a_n |n>$$

$$= \sum_n (\langle n|\psi\rangle) \cdot |n> = \sum_n |n> (\langle n|\psi\rangle) = \sum_n |n\rangle\langle n|\psi\rangle$$

$|\psi\rangle$为任意态矢，求和与之 无关，可得

$\sum_n |n\rangle\langle n| = 1 \Rightarrow$ 本征矢的封闭性，使表 象变换大为简化

$|n\rangle\langle n|$ 和 $|q\rangle\langle q|$可定义为投影算符

$$|n\rangle\langle n|\psi\rangle: \ |\psi\rangle \rightarrow |n\rangle\langle n|\psi\rangle$$

$$|q\rangle\langle q|\psi\rangle: \ |\psi\rangle \rightarrow |q\rangle\langle q|\psi\rangle$$

$\psi(x, t)$ 表示坐标X表象中的状态 $|\psi\rangle$, 则

$$\begin{cases} \langle x|\psi\rangle = \psi(x,t) \\ \langle \psi|x\rangle = \langle x|\psi\rangle^* = \psi^*(x,t) \end{cases}$$

$\langle x|$ ×

×$|x'\rangle$

$$\sum_n |n\rangle\langle n| = 1$$

坐标表象中的封闭性

$$\sum_n \langle x|n\rangle\langle n|x'\rangle = \langle x|x'\rangle \rightarrow \sum_n u_n^*(x')u_n(x) = \delta(x-x')$$

$$\int |q\rangle dq \langle q| = 1$$

$$\int \langle x|q\rangle dq \langle q|x'\rangle = \langle x|x'\rangle \rightarrow \int u_q^*(x')u_q(x)dq = \delta(x-x')$$

正交归一性

（本征值不同）

$$\int u_n^*(x)u_m(x)dx = \delta_{nm}$$

$$\int u_{q'}^*(x)u_q(x)dx = \delta(q-q')$$

$$\sum_n u_n^*(x')u_n(x) = \delta(x-x')$$

$$\int u_q^*(x')u_q(x)dq = \delta(x-x')$$

dx *n* 或 *dq*

本征值既有分立

又有联系谱的封闭性表示

$$\sum_n |n\rangle\langle n| + \int |q\rangle dq \langle q| = 1$$

（3）算符

(a) 右矢

$$\psi(x,t)=\hat{F}(x,\hat{p})\phi(x,t)$$

坐标X表象

$$|\psi>=\hat{F}|\phi>$$

$$<m|\psi>=<m|\hat{F}|\phi> \;=\sum_n <m|\hat{F}|n><n|\phi>$$

F_{mn}----
F在Q表象的矩阵元

$$\sum_n |n><n|=1$$

矩阵

$$\begin{pmatrix} <1|\psi> \\ <2|\psi> \\ \vdots \\ <n|\psi> \\ \vdots \end{pmatrix} = \begin{pmatrix} <1|\hat{F}|1>,<1|\hat{F}|2>,\cdots\cdots \\ <2|\hat{F}|1>,<2|\hat{F}|2>,\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ <n|\hat{F}|1>\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix} \begin{pmatrix} <1|\phi> \\ <2|\phi> \\ \vdots \\ <n|\phi> \\ \vdots \end{pmatrix}$$

Q表象

$$\psi = F\,\varphi$$

$$\overline{F} = <\psi|\hat{F}|\psi> \;\Rightarrow\; \overline{F} = \sum_{mn} <\psi|m><m|\hat{F}|n><n|\psi>$$

$$\uparrow \sum_m |m><m| \qquad \nwarrow \sum_n |n><n|$$

$$= \sum_{mn} a_m^* F_{mn} a_n$$

（b）左矢

$$<\psi|m> = <m|\psi>^* = \left(\sum_n <m|\hat{F}|n><n|\phi>\right)^* = \left(\sum_n F_{mn} <n|\phi>\right)^*$$

$$= \sum_n F_{mn}^* <n|\phi>^* = \sum_n \tilde{F}_{nm}^* <\phi|n> = \sum_n (F^+)_{nm} <\phi|n>$$

$$= \sum_n <\phi|n><n|\hat{F}^+|m> = <\phi|\hat{F}^+|m> \;\Rightarrow\; <\psi| = <\phi|\hat{F}^+$$

若 **F**是厄米算符

$$<\psi| = <\phi|\hat{F}$$

$$<\psi| = <\phi|\hat{F} \qquad \hat{F}|\phi> = |\psi>$$

$$|\psi> = \hat{x}|\phi> \qquad <p|\psi> = <p|\hat{x}|\phi> = \int <p|\hat{x}|p'> dp' <p'|\phi>$$

$$<p|\hat{x}|p'> = \int\int <p|x> dx <x|\hat{x}|x'> dx' <x'|p'>$$

$$= \int\int <p|x> dx x' <x|x'> dx' <x'|p'>$$

$$= \int\int <p|x> dx x' \delta(x-x') dx' <x'|p'>$$

$$= \int\int <p|x> x dx <x|p'> = \frac{1}{2\pi\hbar}\iint e^{-\frac{i}{\hbar}px} x e^{\frac{i}{\hbar}p'x} dx$$

$$= \frac{1}{2\pi\hbar}\iint i\hbar\frac{\partial}{\partial p} e^{-\frac{i}{\hbar}px} e^{\frac{i}{\hbar}p'x} dx = \frac{1}{2\pi\hbar} i\hbar\frac{\partial}{\partial p}\iint e^{-\frac{i}{\hbar}px} e^{\frac{i}{\hbar}p'x} dx$$

$$= i\hbar\frac{\partial}{\partial p}\delta(p-p')$$

$$<p|\psi> = <p|\hat{x}|\phi> = \int <p|\hat{x}|p'> dp' <p'|\phi>$$

$$= \int i\hbar\frac{\partial}{\partial p}\delta(p-p') dp' <p'|\phi> = i\hbar\frac{\partial}{\partial p}<p|\phi> \qquad \hat{x} = i\hbar\frac{\partial}{\partial p}$$

（4）小结

性质	坐标X表象	狄拉克符号
状态波函数	$\Psi(x,t)$	$\vert\Psi>$
算符	$\hat{F}(\vec{r},-i\hbar\nabla)$	$\hat{F}$
归一化	$\int u_m^*(x)u_n(x)dx=\delta_{mn}$	$<m\vert n>=\delta_{mn}$
波函数	$\int\Psi^*(x,t)\Psi(x,t)dx=1$	$<\Psi\vert\Psi>=1$
正交归一性	$\int u_{q'}^*(x)u_{q''}(x)dx=\delta(q'-q'')$	$<q'\vert q''>=\delta(q'-q'')$
本征函数封闭性	$\sum_n u_n(x)u_n^*(x')=\delta(x'-x)$ $\int u_q(x)u_q^*(x')dq=\delta(x'-x)$	$\sum_n\vert n><n\vert=1$ $\int\vert q>dq<q\vert=1$
公式	$\Psi(x,t)=\hat{F}(x,\hat{p}_x)\Phi(x,t)$	$\vert\Psi>=\hat{F}\vert\Phi>$
本征函数	$\hat{F}(\vec{r},\hat{\vec{p}})\psi(\vec{r})=\lambda\psi(\vec{r})$	$\hat{F}\vert\psi>=\lambda\vert\psi>$
平均值	$\overline{F}=\int\psi^*\hat{F}\psi dx$	$\overline{F}=<\psi\vert\hat{F}\vert\psi>$
矩阵元	$F_{mn}=\int\psi_m^*\hat{F}\psi_n dx$	$F_{mn}=<m\vert\hat{F}\vert n>$
薛定谔方程	$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)=\hat{H}(\vec{r},-i\hbar\nabla)\Psi(\vec{r},t)$	$i\hbar\frac{d}{dt}\vert\Psi(t)>=\hat{H}\vert\Psi(t)>$

§4.6　线性谐振子和占有数表象

（1）算符 a, a+, N

（2）占有数表象

（1）算符 a，a+，N

（a）坐标表象中的线性谐振子

$$\begin{cases} \hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \\ \psi_n = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x) & \alpha = \sqrt{\frac{m\omega}{\hbar}} \\ E_n = (n+\frac{1}{2})\hbar\omega & n = 0,1,2\cdots \end{cases}$$

（b）算符 a，a+，N的定义

假定

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}[\hat{x} + \frac{i}{m\omega}\hat{p}] = \frac{\alpha}{\sqrt{2}}[\hat{x} - \frac{1}{i\hbar\alpha^2}\hat{p}]$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}[\hat{x} - \frac{i}{m\omega}\hat{p}] = \frac{\alpha}{\sqrt{2}}[\hat{x} + \frac{1}{i\hbar\alpha^2}\hat{p}]$$

$$[\hat{a},\ \hat{a}^+] = 1$$

证明 $[\hat{a},\ \hat{a}^+]=1$

$$[\hat{a},\ \hat{a}^+]=\left[\frac{\alpha}{\sqrt{2}}(\hat{x}-\tfrac{1}{i\hbar\alpha^2}\hat{p}),\ \frac{\alpha}{\sqrt{2}}(\hat{x}+\tfrac{1}{i\hbar\alpha^2}\hat{p})\right]$$

$$=\frac{\alpha^2}{2}\left[\hat{x}-\tfrac{1}{i\hbar\alpha^2}\hat{p},\ \hat{x}+\tfrac{1}{i\hbar\alpha^2}\hat{p}\right]$$

$$=\frac{\alpha^2}{2}\{[\hat{x},\hat{x}]+[\hat{x},\frac{1}{i\hbar\alpha^2}\hat{p}]-[\frac{1}{i\hbar\alpha^2}\hat{p},\hat{x}]-[\frac{1}{i\hbar\alpha^2}\hat{p},\frac{1}{i\hbar\alpha^2}\hat{p}]\}$$

$$=\frac{\alpha^2}{2}\frac{1}{i\hbar\alpha^2}\{[\hat{x},\hat{p}]-[\hat{p},\hat{x}]\}$$

$$=\frac{\alpha^2}{2}\frac{1}{i\hbar\alpha^2}\{2i\hbar\}$$

$$=1$$

（c）a, a+ 及线性谐振子的哈密顿算符

$$\hat{a} = \frac{\alpha}{\sqrt{2}}[\hat{x} - \tfrac{1}{i\hbar\alpha^2}\hat{p}] \qquad (1)$$

$$\hat{a}^+ = \frac{\alpha}{\sqrt{2}}[\hat{x} + \tfrac{1}{i\hbar\alpha^2}\hat{p}] \qquad (2)$$

$$\Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}\alpha}[(2)+(1)] = \frac{1}{\alpha\sqrt{2}}[a^+ + a] \\ \hat{p} = i\hbar\frac{\alpha}{\sqrt{2}}[(2)-(1)] = i\hbar\frac{\alpha}{\sqrt{2}}[a^+ - a] \end{cases}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2m} i\hbar\tfrac{\alpha}{\sqrt{2}}[\hat{a}^+ - \hat{a}] i\hbar\tfrac{\alpha}{\sqrt{2}}[\hat{a}^+ - \hat{a}] + \frac{1}{2}m\omega^2 \tfrac{1}{\alpha\sqrt{2}}[\hat{a}^+ + \hat{a}]\tfrac{1}{\alpha\sqrt{2}}[\hat{a}^+ + \hat{a}]$$

$\alpha^2 = m\omega/\hbar$

$$= -\frac{\alpha^2\hbar^2}{4m}[\hat{a}^+\hat{a}^+ + \hat{a}\hat{a} - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+] + \frac{1}{4\alpha^2}m\omega^2[\hat{a}^+\hat{a}^+ + \hat{a}\hat{a} + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+]$$

$$= -\frac{\hbar\omega}{4}[\hat{a}^+\hat{a}^+ + \hat{a}\hat{a} - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+] + \frac{1}{4}\hbar\omega[\hat{a}^+\hat{a}^+ + \hat{a}\hat{a} + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+]$$

$$= \frac{1}{2}\hbar\omega[\hat{a}^+\hat{a} + \hat{a}\hat{a}^+]$$

$[\hat{a} \quad \hat{a}^+] = 1$

$$= \frac{1}{2}\hbar\omega[\hat{a}^+\hat{a} + \hat{a}^+\hat{a} + 1] = \hbar\omega[\hat{a}^+\hat{a} + \tfrac{1}{2}] = \hbar\omega[\hat{N} + \tfrac{1}{2}]$$

$\hat{N} = \hat{a}^+\hat{a}$ 为粒子数算符

（d） 算符a, a+, N 的物理意义

d.I a, a+的物理意义

$$\hat{a}=\frac{\alpha}{\sqrt{2}}[\hat{x}-\frac{1}{i\hbar\alpha^2}\hat{p}]=\frac{\alpha}{\sqrt{2}}[\hat{x}+\frac{1}{\alpha^2}\frac{\partial}{\partial x}]=\frac{\alpha}{\sqrt{2}}x+\frac{1}{\alpha\sqrt{2}}\frac{\partial}{\partial x}=\frac{1}{\sqrt{2}}(\xi+\frac{\partial}{\partial\xi})$$

$$\hat{a}^{+}=\frac{\alpha}{\sqrt{2}}[\hat{x}+\frac{1}{i\hbar\alpha^2}\hat{p}]=\frac{\alpha}{\sqrt{2}}x-\frac{1}{\alpha\sqrt{2}}\frac{\partial}{\partial x}=\frac{1}{\sqrt{2}}(\xi-\frac{\partial}{\partial\xi}),\quad \xi=\alpha x$$

$$\hat{a}\psi_n=\frac{1}{\sqrt{2}}(\xi+\frac{\partial}{\partial\xi})\psi_n$$
$$=\sqrt{n}\psi_{n-1}$$

$$\xi\psi_n(\xi)=\sqrt{\frac{n}{2}}\psi_{n-1}(\xi)+\sqrt{\frac{n+1}{2}}\psi_{n+1}(\xi)$$

$$\frac{d}{d\xi}\psi_n(\xi)=\sqrt{\frac{n}{2}}\psi_{n-1}(\xi)-\sqrt{\frac{n+1}{2}}\psi_{n+1}(\xi)$$

同理， $\hat{a}^{+}\psi_n=\sqrt{n+1}\psi_{n+1}$

粒子湮灭

$$\hat{a}|n>=\sqrt{n}|n-1>$$

粒子产生

$$\hat{a}^{+}|n>=\sqrt{n+1}|n+1>$$

|n>, |n-1>, |n+1> 是H 的本征基矢， E_n, E_{n-1}, E_{n+1} 是本征值.

振子能量--ħω---声子
状态 |n >---第 n 声子态

且 $\hat{a}|0>=0$

基态

$$\hat{a}^+|0>=\sqrt{0+1}|0+1> \quad \rightarrow \quad |1>=\frac{1}{\sqrt{1}}\hat{a}^+|0>$$

$$\hat{a}^+|1>=\sqrt{1+1}|1+1> \rightarrow |2>=\frac{1}{\sqrt{2}}\hat{a}^+|1>=\frac{1}{\sqrt{2}}\frac{1}{\sqrt{1}}\hat{a}^+\hat{a}^+|0>=\frac{1}{\sqrt{2!}}(\hat{a}^+)^2|0>$$

$$\cdots\cdots\cdots\cdots \quad |n>=\frac{1}{\sqrt{n!}}(\hat{a}^+)^n|0>$$

d.II 算符N的意义

$$\hat{N}|n>=\hat{a}^+\hat{a}|n> \quad =\hat{a}^+\sqrt{n}|n-1>$$

$$=\sqrt{n}\sqrt{(n-1)+1}|n>$$

$$=n|n>$$

n 是算符N的本征值，描述粒子数目，定义为粒子数算符

（2） 占有数表象

基于|n > 的表象称为占有数表象

算符a的矩阵元 $< n' | \hat{a} | n > = \sqrt{n} < n' | n-1 > = \sqrt{n}\delta_{n'n-1}$

算符$\mathbf{a}^{+}$的矩阵元 $< n' | \hat{a}^{+} | n > = \sqrt{n+1} < n' | n+1 > = \sqrt{n+1}\delta_{n'n+1}$

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \qquad a^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$